Elastic string in a random medium

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(Received 27 November 1995)

We consider a one-dimensional elastic string as a set of massless beads interacting through springs characterized by anisotropic elastic constants. The string, driven by an external force, moves in a medium with quenched disorder. We find that longitudinal fluctuations lead to nonlinear behavior in the equation of motion that is *kinematically* generated by the motion of the string. The strength of the nonlinear effects depends on the anisotropy of the medium and the distance from the depinning transition. On the other hand, the consideration of restricted solid-on-solid conditions imposed on the string leads to a nonlinear term with a *diverging* coefficient at the depinning transition.

PACS number(s): 47.55.−t

The motion of an elastic string in disordered media has attracted considerable attention recently, in part due to its relevance to flux flow in type-II superconductors [1] and roughening of nonequilibrium interfaces [2]. By means of a number of numerical [3–8] and analytical [9,10] studies it has been observed that scaling theory can be used as an underlying framework to understand and characterize the dynamical properties of the elastic string.

Consider a one-dimensional elastic string moving under the influence of an external driving force F normal to the string, in a two-dimensional disordered medium of horizontal size L (along the x axis). A discrete model for such a string consists on N massless beads connected by springs. The string is assumed to be oriented along the x axis and the position of the ith bead is denoted by a two-dimensional displacement vector $\vec{r}_i \equiv (x_i, y_i)$, $i=1,\ldots,L$ (see Fig. 1). The disorder in the medium is introduced by uniformly distributed pinning sites with random strength, which we refer to as quenched disorder or ''quenched noise.'' The dynamics of such a string is the result of the interplay between the quenched disorder characteristic of the medium and the elastic properties of the string.

A key quantity is the average velocity of the string as a function of the external force. At small forces F the string is pinned by static disorder. Just above the depinning transition $F=F_c$, i.e., when the external force overcomes the pinning effect of impurities, the velocity varies as

$$v_0(f) \sim f^{\theta}, \tag{1}$$

where θ is the velocity exponent and $f \equiv F/F_c - 1$ the reduced force.

Neglecting thermal fluctuations and lateral fluctuations of the beads, the equation of motion for the string in the continuum limit is the Edwards-Wilkinson equation [11] with quenched disorder [3–10]

$$\frac{\partial y(x,t)}{\partial t} = \nu \nabla^2 y + \eta(x,y) + F. \tag{2}$$

The first term in the right-hand side of (2) includes the elastic effects acting to make the string straight. The second term mimics the quenched disorder, which has zero mean and is uncorrelated. The string is driven in the y direction by the external force F. For large driving force ($F \gg F_c$), the quenched noise becomes effectively time dependent, $\eta(x, y_0 + vt)$. It is believed [10,12] that in this regime the motion of the string induces an additional nonlinear term in (2), namely, the Kardar-Parisi-Zhang (KPZ) term $\lambda(\nabla y)^2$ [13]. However, since this nonlinear term is generated by the motion of the string, λ is expected to vanish as the velocity goes to zero at the depinning transition, and the critical behavior at $F=F_c$ is correctly described by Eq. (2).

While (2) can be obtained (using $\partial_t y = -\delta\mathcal{H}/\delta y + F$) from the Hamiltonian

$$\mathcal{H} = \int_0^L dx \{\nu(\nabla y)^2 + \mu(x,y)\}, \tag{3}$$

the KPZ nonlinear term $\lambda(\nabla y)^2$ cannot be deduced as a variation of any bounded Hamiltonian. Here the quenched noise is $\eta(x,y) = -\delta_y \mu(x,y)$.

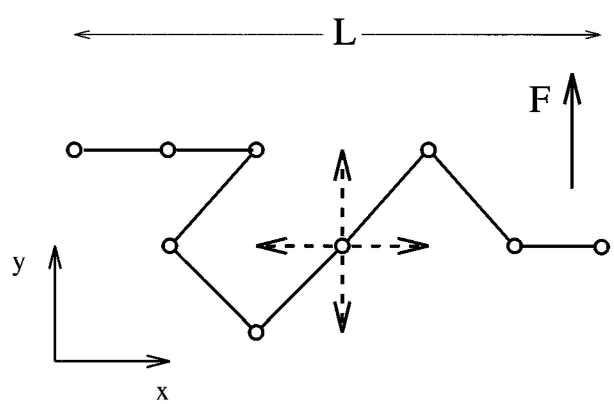

FIG. 1. The discrete version of the elastic string is composed of L massless beads interacting via springs. A driving external force F acts in the y direction. Pointlike quenched disorder (not shown) is introduced at each site on the lattice. The beads are allowed to move in the x and y directions and therefore they develop overhangs.

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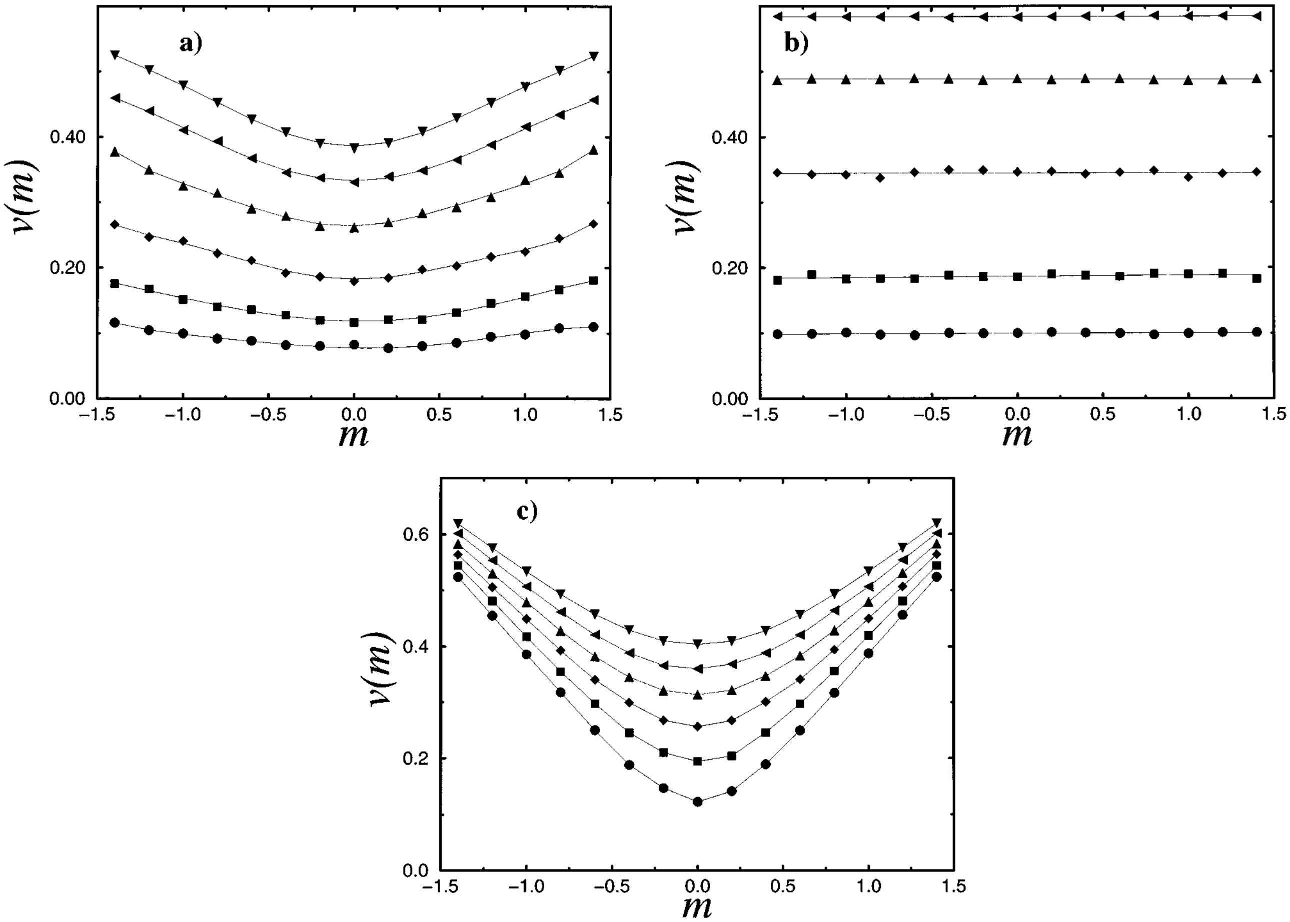

FIG. 2. Plot of average velocity vs the average tilt of the string for different values of the reduced force ranging from $f=0.03$ (bottom curves) to $f=0.20$ (top curves). Results are averaged over 200 independent realizations of the disorder. The system size is $L=250$ and the strength of the disorder is $\delta=3$. (a) The elastic constants are $\nu_x=0.1$ and $\nu_y=1$ ($\varepsilon=10$). The opening of the parabolas as the depinning transition is approached indicates that a nonlinear term is present in the equation of motion, and that its value converges to zero at the depinning transition. The continuous lines are the best polynomial fits to the curves. We note that the observed parabolic dependence cannot be a lattice effect since in this case one expects $v(m=0)=v(m=1)$. (b) The same plot for the case when longitudinal fluctuations are not energetically favorable, so that overhangs are not observed. The elastic constants are $\nu_x=1.0$ and $\nu_y=0.1$ ($\varepsilon=0.1$). The horizontal lines indicate that the velocity is independent of the average tilt of the string. (c) Tilt dependence of the average velocity of the string when the RSOS condition is applied showing the closing of the parabolas indicating a diverging λ term at the depinning transition. Here is shown the isotropic case $\nu_x=\nu_y=1$ ($\varepsilon=1$), although the divergency of λ is shown to be independent of the parameter ε.

In the Hamiltonian (3), only transverse fluctuations (along the y direction) contribute to the elastic energy, forbidding longitudinal fluctuations (along the x direction). However, for a real elastic string, the elastic energy depends on the distance $(\vec{r}_i-\vec{r}_{i-1})^2$ between two consecutives beads. Here we introduce a (1+1)-dimensional model that allows for both longitudinal and transverse fluctuations of the beads. In the model, the elastic energy depends on both $\nu_x(x_i-x_{i-1})^2$ and $\nu_y(y_i-y_{i-1})^2$, where ν_x and ν_y are the elastic constants corresponding to displacements in the x and y directions, respectively. We focus on the determination of the equation of motion of the string. We find that, even though the string can form overhangs, at large enough length scales the string still has a well-defined orientation and profile, and can be described by a continuum theory. The main results of this paper are as follows:

(a) In the limit $\varepsilon\equiv\nu_y/\nu_x\gg1$, where ε is the anisotropy parameter, the large-scale behavior of the string is described by the nonlinear equation of motion with quenched noise

$$\frac{\partial y(x,t)}{\partial t}=\nu\nabla^2 y+\lambda(\nabla y)^2+\eta(x,y)+F, \qquad (4)$$

where the nonlinear term $\lambda(\nabla y)^2$ in (4) is of kinematic origin. We find that λ vanishes at the depinning transition as

$$\lambda(f)\sim f^{|\phi|}\to 0. \qquad (5)$$

(b) If longitudinal fluctuations are neglected, we find numerically that nonlinear terms of the type $\lambda(\nabla y)^2$ are forbidden in the growth equation. We argue that this result applies to a number of previously introduced models [3–8]. In our model, this limit corresponds to $\varepsilon\equiv\nu_y/\nu_x\to0$.

(c) A different scenario is found when the rules of motion of the beads are constrainted to satisfy a restricted solid-on-solid (RSOS) condition $|h_{i\pm1}-h_i|\leqslant$const [14]. When such a condition is imposed we find that the equation of motion of the string is (4) but with a coefficient λ that diverges at the depinning transition as

$$\lambda(f) \sim f^{-\phi} \rightarrow \infty. \tag{6}$$

This result is valid for any value of the anisotropy parameter ε, and applies to a number of growth models in the directed percolation universality class [15–20].

We now take up each of these results in turn. Before beginning, we note that for a given model the presence of a nonlinear term $\lambda(\nabla y)^2$ can be identified using tilt-dependent velocity measurements [21,18–20]. Suppose we tilt the elastic string, by imposing helical boundary conditions $y_1 = y_L + mL$, where m is the average tilt of the string. Then, according to (4), the average tilt-dependent velocity becomes

$$v(m) = v_0 + \lambda m^2, \tag{7}$$

where v_0 is the velocity of the untilted string. If $\lambda = 0$, so that the motion of the elastic string is described by (2), then the velocity does not depend on the average tilt of the interface. Tilt dependence is expected only if there is a nonlinear term in the equation of motion of the form $\lambda(\nabla y)^2$. This property can be used to gain information on the presence and magnitude of the nonlinear term λ, by monitoring the velocity of the string as a function of the average tilt, and fitting to a parabola the obtained curve [22].

In the following, we study a generalized model of an elastic string that allows for lateral motions of the beads and therefore overhangs. The main element of the model, not included in the Hamiltonian (3), is the existence of longitudinal motion of the beads. To include this additional degree of freedom, we use a generalized Hamiltonian

$$\mathscr{H} = \sum_{i=1}^{L} [\nu_x(x_i - x_{i-1})^2 + \nu_y(y_i - y_{i-1})^2 + \mu(x_i, y_i) - F y_i]. \tag{8}$$

We simulate the discrete version of (8), concentrating on the zero-temperature dynamics of the string (only motions that decrease the total energy of the string are allowed). A standard Monte Carlo algorithm, by choosing randomly a site on the interface, induces time-dependent noise. Since at zero temperature the motion of the string is deterministic, we have employed an algorithm with parallel updating, during which even and odd sublattices are updated simultaneously. The quenched noise is introduced by defining at every site of the two-dimensional lattice uncorrelated random numbers $\mu(i,j)$, uniformly distributed between $-\delta$ and δ.

During the simulations, the chosen bead is allowed to move to one of its *four* nearest neighbors, if that motion decreases the total energy of the string given by (8). If there is more than one possible move with $\Delta\mathscr{H} < 0$, then the one with most negative $\Delta\mathscr{H}$ is chosen. We focus on the determination of the nonlinear term λ, measuring the tilt-dependent velocity of the string.

(a) Figure 2(a) shows the velocity of the driven elastic string as a function of the average tilt for different driving forces. The results correspond to the anisotropic motion characterized by $\nu_x = 0.1$ and $\nu_y = 1$ ($\varepsilon = 10$), and disorder strength $\delta = 3$. We see that the velocity follows a parabola with the tilt, indicating the presence of a nonlinear term $\lambda(\nabla y)^2$ above the depinning transition (moving phase, $F > F_c$). However, the parabolas become flatter as the depinning transition is approached. Our calculations indicate that $\lambda \rightarrow 0$ as $F \rightarrow F_c$ as in (5). These results are obtained for the anisotropic case $\nu_x < \nu_y$ ($\varepsilon > 1$), and further increasing the anisotropy, the observed behavior does not vanish.

(b) The other limit of the model leads to known results: a finite ν_y and $\nu_x \rightarrow \infty$ means that longitudinal fluctuations are energetically very expensive, allowing only transversal fluctuations. In this limit, the model reduces to the models of Refs. [3–8], where longitudinal fluctuations are not allowed. In this case the nonlinear term is *exactly* zero [see Fig. 2(b)]. Thus as $\nu_x \rightarrow \infty$, a decrease of λ toward zero is expected.

Our simulations at the isotropic point $\varepsilon = 1$ ($\nu_x = \nu_y = 1$) show the following results: (i) for large disorder strength ($\delta = 3 > \nu_x = \nu_y = 1$) we find a coefficient $\lambda \rightarrow 0$ as $F \rightarrow F_c$; (ii) for disorder strength $\delta = 1 = \nu_x = \nu_y$ we find $\lambda = 0$ for any value of the force. Thus, there are two scenarios compatible with our results. According to the first, $\lambda \rightarrow 0$ as $\varepsilon \rightarrow 0$ (strong disorder). The second scenario (small disorder) says that $\lambda = 0$ for $\varepsilon \leqslant 1$ and $\lambda \neq 0$ for $\varepsilon > 1$.

(c) Figure 2(c) shows the results of our simulations when the RSOS condition is applied to the growth of the string: for a given ith bead, if $h_{i\pm1} - h_i > 2$ then we increment $h_i \rightarrow h_i + 1$ regardless of the energy value of the new configuration. In contrast with the results of (a) and (b), in this case we find that the parabolas become steeper as $F \rightarrow F_c$, corresponding to an increase in λ as the depinning transition is approached as in Eq. (6) [18–20].

A typical system to which this study may be relevant is the motion of a single flux line in a type-II superconductor, directed along the external magnetic field H [1]. At moderate fields, when the separation of the vortex lines is sufficiently large, the intervortex interaction can be neglected. In this regime, the dynamics of the vortex phase can be understood by studying the motion of a single vortex. We argue that the condition $\nu_x < \nu_y$ can be met in some anisotropic superconductors. Thus our results might be important in understanding the driven diffusion of the flux line. The variation of the velocity with tilt suggests that a tilted external magnetic field H changes the velocity of the flux line, the effect decreasing as we approach the depinning transition.

In summary, we present a model to describe the motion of an elastic anisotropic string in a disordered medium. We find that when transverse fluctuations (along the driving force) are energetically more favorable than longitudinal fluctuations, the string is described by Eq. (2) not only at the depinning transition but only in the moving phase. However, if longitudinal fluctuations are more favorable, a kinematic nonlinear term is induced so that its coefficient vanishes as we approach the depinning transition. The directed percolation depinning universality class is obtained when a RSOS condition is applied that favors the growth of regions with large local slopes. This last result is shown to be valid for any value of the anisotropic parameter ε.

We thank L. A. N. Amaral, R. Cuerno, K. B. Lauritsen, and S. Tomassone for valuable discussions. The Center for Polymer Studies is supported by the NSF.

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